A new algebraic analysis of linear mixed models

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Abstract

This article presents a new investigation to the linear mixed model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}$ with fixed effect $\mathbf{X}\boldsymbol{\beta}$ and random effect $\mathbf{Z}\boldsymbol{\gamma}$ under a general assumption via some novel algebraic tools in matrix theory, and reveals a variety of deep and profound properties hidden behind the linear mixed model. We first derive exact formulas for calculating the best linear unbiased predictor (BLUP) of a general vector $\boldsymbol{\phi} = \mathbf{F}\boldsymbol{\beta} + \mathbf{G}\boldsymbol{\gamma} + \mathbf{H}\boldsymbol{\varepsilon}$ of all unknown parameters in the model by solving a constrained quadratic matrix-valued function optimization problem in the Löwner partial ordering. We then consider some special cases of the BLUP for different choices of $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ in $\boldsymbol{\phi}$, and establish some fundamental decomposition equalities for the observed random vector $\mathbf{y}$ and its covariance matrix.

Keywords

linear mixed model, fixed effect, random effect, BLUP, BLUE, covariance matrix, decomposition.